BYJU'S Classes

M A T H E M A T I C S

FUNDAMENTALS OF MATHEMATICS

LAWS OF ALGEBRA OF SETS AND FORMULAS BASED ON CARDINALITY OF SETS

NOTES

What you already know

- Subsets and their type
- Power set
- Venn diagram
- Operations of sets

What you will learn

- Symmetric difference of set
- Complement of set
- Laws and algebra of set
- De Morgan's law

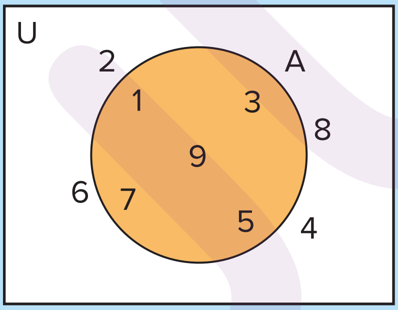
Complement of a set

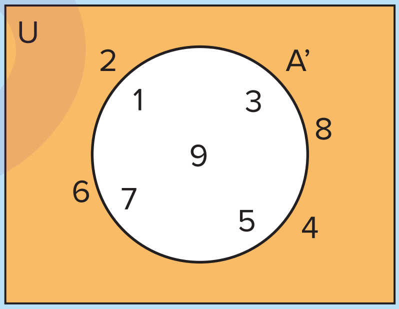
Complement of a set A represents the negation of the set A, which means everything except the set A. It is denoted by A^c or A'.
$A^c = U - A$, where U is the universal set.

Example

Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9}
and A = {1, 3, 5, 7, 9}.
Find A^c.

$A^c = \{2, 4, 6, 8\}$





Complement of a set

$A \cap A^c = \Phi$

$A \cup A^c = U$

$(A^c)^c = A$

$U^c = \Phi$

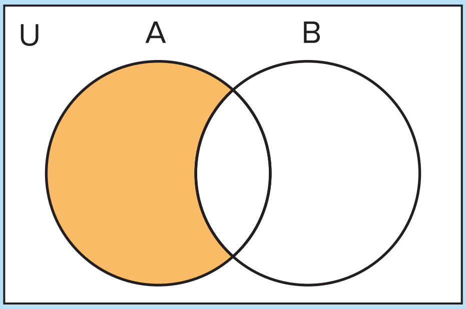
$\Phi^c = U$

Difference of sets

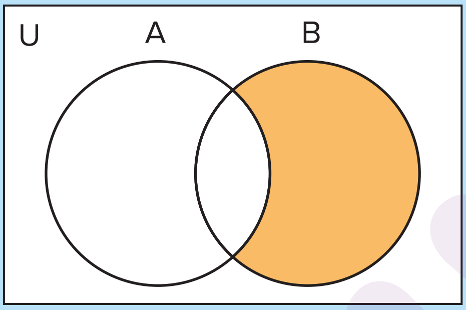
The difference of A and B, i.e., A − B = {All those elements of A that do not belong to B}.
Similarly, B − A = {All those elements of B that do not belong to A}

In other words, the difference of the sets A and B in this order is the set of elements that belong to A but not to B

A - B



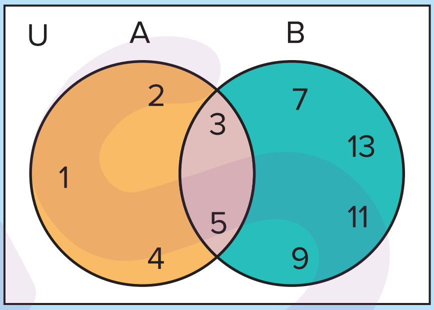
B - A



Example

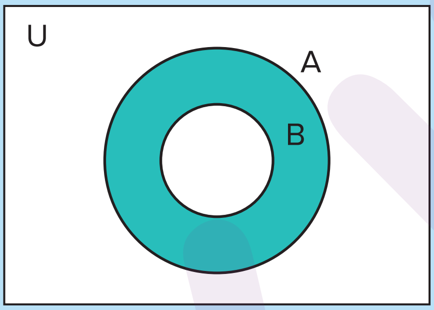
A = {1, 2, 3, 4, 5},
B = {3, 5, 7, 9, 11, 13}

A - B

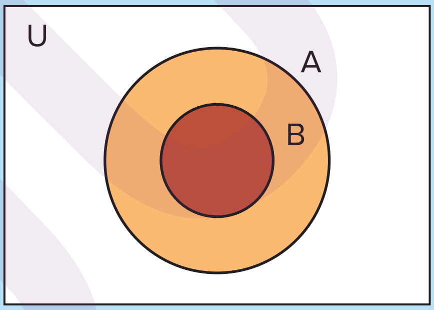


A - B = {1, 2, 4},
B - A = {7, 9, 11, 13}

Observation I: B is a subset of A

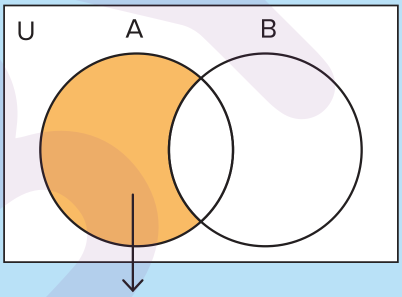


A - B

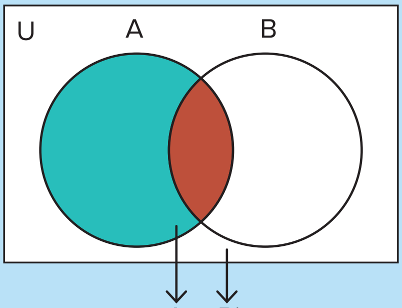


B - A = Φ

Observation II: *A* and *B* are not equal sets but they have some elements in common



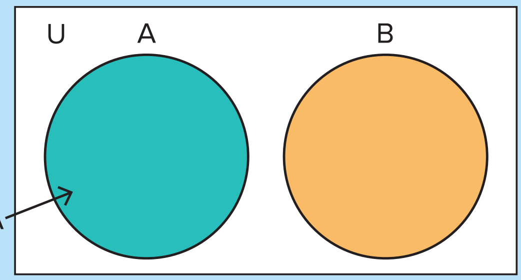
A - B

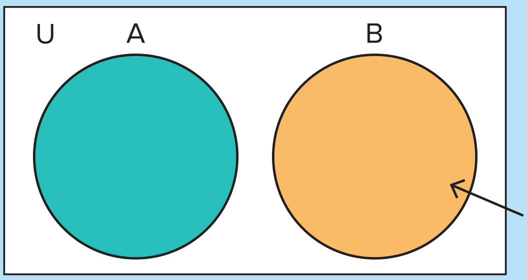


A ∩ B'

Observation III: *A* and *B* have no elements in common.

A - B = A





B - A = B

Mathematically

A − B = {x : x ∈ A and x ∉ B}
Thus, x ∈ (A − B) ⟺ x ∈ A and x ∉ B
Also, (B − A) = {x : x ∈ B and x ∉ A}
Thus, x ∈ (B − A) ⟺ x ∈ B and x ∉ A

Symmetric difference of sets

Symmetric difference of sets A and B is the set of all the elements that are either in A or in B but not in both.

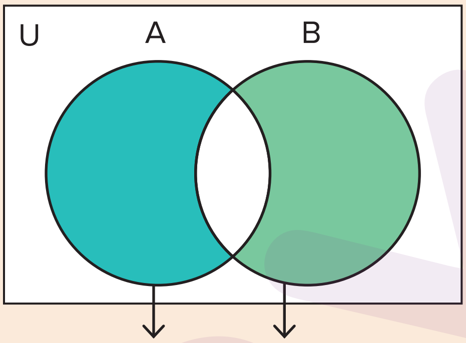
Notation Symmetric difference of sets A and B : A Δ B

A Δ B = {All those elements that belong in either A − B or in B − A}

A Δ B = (A − B) ∪ (B − A)

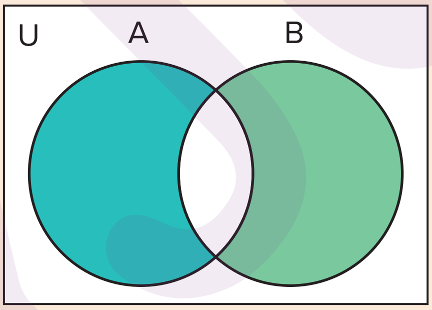
Observation

1. A Δ B = (A − B) ∪ (B − A)

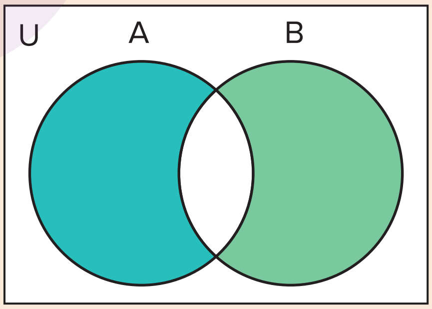


A Δ B = (A − B) ∪ (B − A)

2. A Δ B = (A ∪ B) − (A ∩ B)



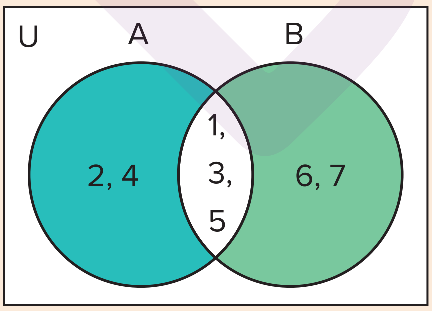
A Δ B = (A − B) ∪ (B − A)



A Δ B = (A ∪ B) − (A ∩ B)

Example

Let us consider A = {1, 2, 3, 4, 5}, B = {1, 3, 5, 6, 7}



A Δ B = (A − B) ∪ (B − A) = {2, 4, 6, 7}

Mathematically

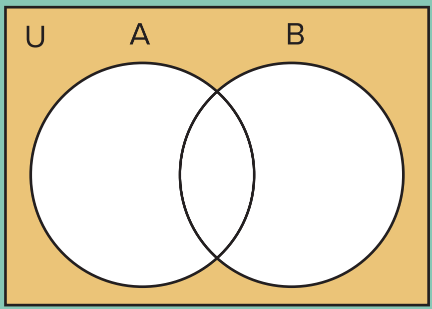
A Δ B = (A − B) ∪ (B − A)
= {x : x ∈ (A − B) or (B − A)}

Properties of symmetric difference

- A Δ A = Φ
- A Δ Φ = A
- A Δ B = B Δ A (Commutative law)
- If A Δ B = A Δ C, then B = C
- A Δ (B Δ C) = (A Δ B) Δ C (Associative law)
- A ∩ (B Δ C) = (A ∩ B) Δ (A ∩ C) (Distributive law)

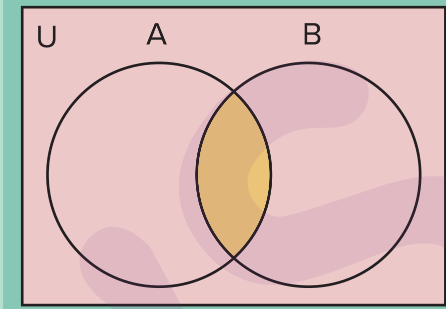
De Morgan's law

$(A \cup B)^c = A^c \cap B^c$



$A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$



$A^c \cup B^c$

Example

If $A = \{1, 2\}$, $B = \{2, 3\}$, and $U = \{1, 2, 3, 4\}$, then prove De Morgan's law.

$(A \cup B)^c = A^c \cap B^c$

$A \cup B = \{1, 2, 3\}$

$(A \cup B)^c = \{4\}$

$A^c = \{3, 4\}$

$B^c = \{1, 4\}$

Therefore,

$A^c \cap B^c = \{4\}$

Hence proved.

Similarly,

$(A \cap B)^c = A^c \cup B^c$

$(A \cap B)^c = \{1, 3, 4\}$ and

$A^c \cup B^c = \{3, 4\} \cup \{1, 4\}$
$= \{1, 3, 4\}$

Hence proved.

$A = \{x: x = 2n, n \leq 3, n \in N\}$ and
$B = \{x: x = 3n, n \leq 3, n \in N\}$.
Find the following:

1. $A \cup B$
2. $A \cap B$
3. $A - B$
4. $B - A$
5. $A \Delta B$

Solution

$A = \{2, 4, 6\}$

$B = \{3, 6, 9\}$

1. $A \cup B = \{2, 3, 4, 6, 9\}$
2. $A \cap B = \{6\}$
3. $A - B = \{2, 4\}$
4. $B - A = \{3, 9\}$
5. $A \Delta B = (A - B) \cup (B - A) = \{2, 3, 4, 9\}$

If $X = \{4^n - 3n - 1, n \in N\}$ and $Y = \{9(n - 1), n \in N\}$, where N is a set of natural numbers, then find $X \cup Y$.

1. X
2. Y
3. N
4. Y - X

Solution

Given, $X = \{4^n - 3n - 1, n \in N\}$
Putting some values of natural numbers, we get, $X = \{0, 9, 54, 243, \ldots\}$
Similarly, $Y = \{0, 9, 18, 27, 36, 45, 54, \ldots\}$
It is clear that both X and Y contain multiples of 9 and Y is the superset of X, i.e., every element of X belongs to the set Y. Hence $X \cup Y$ will be the bigger set, Y.

Formula for cardinality

Type 1 $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A and B are disjoint non-void sets.

Type 2	$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ $n(A - B) = n(A) - n(A \cap B)$ $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
Type 3	$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
Type 4	**Number of elements in exactly two of the sets in** A, B, C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
Type 5	**Number of elements in exactly one of the sets in** A, B, C $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(C \cap A) + 3n(A \cap B \cap C)$ $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$ $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

X and Y are two sets such that X ∪ Y has 18 elements, X has 12 elements, and Y has 15 elements. Find the number of elements in X ∩ Y.

Solution

We know that, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Here, $n(X \cup Y) = 18$, $n(X) = 12$, and $n(Y) = 15$

Putting the values in the formula, we get,

$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$18 = 12 + 15 - n(X \cap Y) \Rightarrow n(X \cap Y) = 9$

A class has 175 students. The following data shows the number of students opting one or more subjects: Mathematics - 100, Physics - 70, Chemistry - 40, Mathematics and Physics - 30, Mathematics and Chemistry - 28, Physics and Chemistry - 23, Mathematics, Physics, and Chemistry - 18. How many students have opted for Mathematics alone?

Solution

Total students = 175
Mathematics (M) = 100
Physics (P) = 70
Chemistry (C) = 40
$n(M \cap P) = 30$
$n(M \cap C) = 28$
$n(P \cap C) = 23$
$n(M \cap P \cap C) = 18$

Here, $e = n(M \cap P \cap C) = 18$

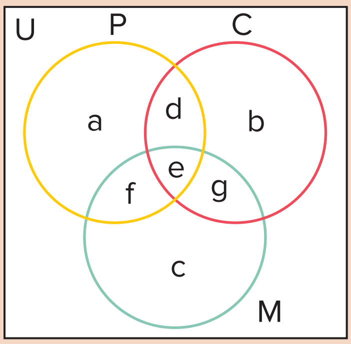
$d + e = n(P \cap C) = 23;\ d = 5$

$e + g = n(M \cap C) = 28;\ g = 10$

$e + f = n(P \cap M) = 30;\ f = 12$

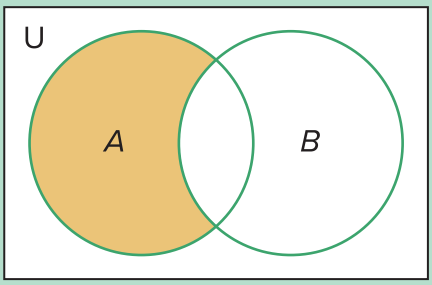
$e + f + g + c = n(M) = 100$

$\Rightarrow c = 100 - (18 + 10 + 12) = 60$

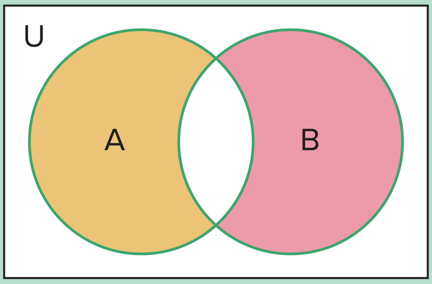


Observation

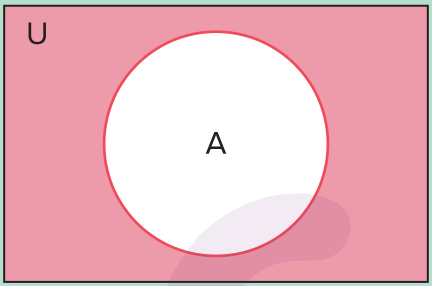
$n(A - B) = n(A) - n(A \cap B)$



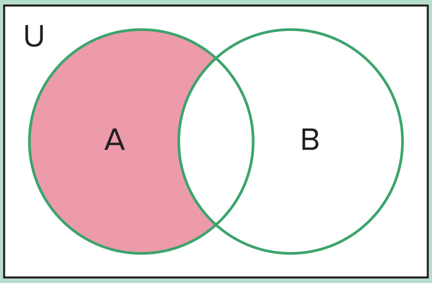
$n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$



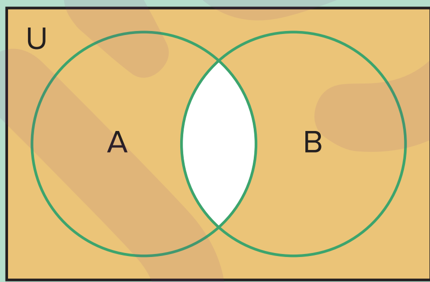
$n(A') = n(U) - n(A)$



$n(A \cap B') = n(A) - n(A \cap B)$



$n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$



Concept Check

1. If A and B are two sets, then find $(A \cup B)' \cup (A' \cap B)$.
   (a) A' (b) B' (c) A (d) None
2. If A and B are two sets, then find $A \cap (A \cup B)$.
   (a) A (b) B (c) Φ (d) $A \cup B$
3. In a class of 60 students, 25 students play cricket, 20 students play tennis, and 10 play both the games. Find the number of students who neither play cricket nor tennis.

Summary sheet

Key Takeaways

- Symmetric difference of two sets A and B is denoted by $A \Delta B = (A - B) \cup (B - A)$
- The commutative property for the union and intersection says that the order of sets in which we do the operation does not change the result. For example, $A \cup B = B \cup A$ and $A \cap B = B \cap A$
- The associative property for the union and intersection says that the way sets are grouped does not change the result. For example, $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
- The distributive property of union over intersection and intersection over union show two ways of finding results for certain problems by mixing the set operations of union and intersection. For example, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's law states that the complement of the union of two sets is the intersection of their complements and the complement of the intersection of two sets is the union of their complements, i.e., $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

Key Formulae

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
- Number of elements in exactly two of the sets in A, B, C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
- Number of elements in exactly one of the sets in A, B, C $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(C \cap A) + 3n(A \cap B \cap C)$
- $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
- $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Mind Map

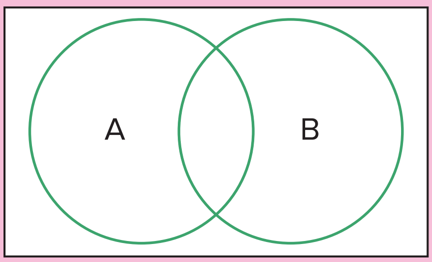
Algebra of set

- Complement of set
- Difference of set
- Symmetrical difference of set
- Properties and law of set
- De morgan's law
- Cardinality formulae
- Observations using venn diagram

Self-Assessment

1. In the given diagram, $n(A) = 15$, $n(B) = 25$, $n(A \cap B) = 5$, and $n(E) = 50$
   (a) Insert the number of elements into each of the four regions.
   (b) Find $n(A \cup B)$ and $n(A \cap B')$.

E



2. If a(t), b(t), and c(t) are the lengths of the three sides of a triangle t in a non-decreasing order, i.e., $a(t) \le b(t) \le c(t)$, we define the sets as:
   - X = {Triangle t: a(t) = b(t)}
   - Y = {Triangle t: b(t) = c(t)}
   - T = The set of all triangles

   Using only set operations on these three sets, define the following:
   (a) The set of all equilateral triangles (all sides are equal)
   (b) The set of all isosceles triangles (at least two sides are equal)
   (c) The set of all scalene triangles (no two sides are equal)

3. Let A = {1, 3, 5, 7, 9, 11, 13} and B = {x: x = 3n, n < 5, n ∈ N }. Find A Δ B

Answers

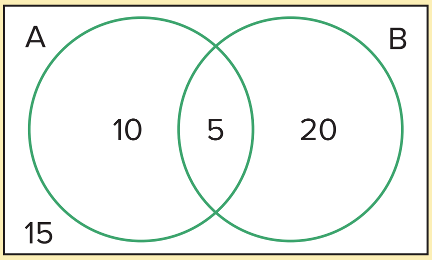
Concept Check

1. (A U B)’ = A’ ∩ B’
   So, (A U B)’ U (A’ ∩ B) can be written as (A’ ∩ B’) U (A’ ∩ B).
   (A’ ∩ B’) U (A’ ∩ B) = A’ ∩ (B U B’); (using distributive property, A ∩ (B U C) = (A ∩ B) U (A ∩ C))
   A’ ∩ (B U B’) = A’ ∩ U = A’

2. We know that set operations are commutative, hence
   A ∩ (A U B) = (A ∩ A) U (A ∩ B) = A U (A ∩ B) (Idempotent law: (A ∩ A) = A)
   (A ∩ B) is the subset of A, hence A U (A ∩ B) = A; ((Subset) U (Superset) = Superset)

3. Total students = 60
   Cricket (n(C)) = 25
   Tennis (n(T)) = 20
   Students playing both cricket and tennis = n(C ∩ T) =10
   Students playing either cricket or tennis = n(C U T) = n(C) + n(T) - n(C ∩ T);
   {n(X ∪ Y) = n(X) + n(Y) − n(X ∩ Y)}
   ⇒ n(C U T) = 25 + 20 − 10 = 35
   Number of students who neither play cricket nor tennis = Total students − Students playing either cricket or tennis = 60 − 35 = 25

Self-Assessment



1. (a) We begin at the intersection and work outwards.
   The intersection A ∩ B has 5 elements
   Hence, the region of A outside A ∩ B has 10 elements,
   and the region of B outside A ∩ B has 20 elements.
   This makes 35 elements so far, so the outer region has 15 elements.
   (b) From the diagram, |A ∪ B| = 35 and |A ∩ B’| = 10

2. (a) Clearly, for equilateral triangles, a(t) = b(t) = c(t), therefore, the answer is X ∩ Y.
   (b) For isosceles triangles, either a(t) = b(t) or b(t) = c(t), so the required set is X ∪ Y.
   (c) A triangle is scalene if and only if it is not isosceles. Using the result of the previous part, the set of scalene triangles is T − (X ∪ Y).

3. Given, A = {1, 3, 5, 7, 9, 11, 13} and B = {x: x = 3n, n < 5, n ∈ N}
   B = {3, 6, 9, 12}
   A Δ B = (A − B) U (B − A) = {1, 5, 7, 11, 13, 6, 12}