BYJU'S Classes

M A T H E M A T I C S

FUNDAMENTALS OF MATHEMATICS

MODULUS INEQUALITY

NOTES

What you already know

- Plotting on number line
- Wavy curve method
- Algebraic and geometric interpretation of modulus

What you will learn

- Properties of modulus

We have studied the wavy curve method and number line. In this session, we will be studying the properties of modulus, and how the wavy curve method and the number line plotting are useful in solving questions related to modulus inequalities.

Modulus Inequality (Recall)

Property 1

If $|x| < a \ \forall \ a > 0$
$\Rightarrow -a < x < a$
If $|x| \le a \ \forall \ a > 0$
$\Rightarrow -a \le x \le a$
If $|x| < a \ \forall \ a < 0$
$\Rightarrow x \in \emptyset$

Property 2

If $|x| > a \ \forall \ a > 0$
$\Rightarrow x < -a$ or $x > a$
$\Rightarrow x \in (-\infty, -a) \cup (a, \infty)$
If $|x| \ge a \ \forall \ a > 0$
$\Rightarrow x \le -a$ or $x \ge a$
$\Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

Point to remember:
If $|x| > a \ \forall \ a < 0$
$\Rightarrow x \in R$

Property 3

If $a < |x| < b \ \forall \ a, b > 0$
$\Rightarrow x \in (-b, -a) \cup (a, b)$
If $a \le |x| \le b \ \forall \ a, b > 0$
$\Rightarrow x \in [-b, -a] \cup [a, b]$

Bhaala method (for two-bracket inequalities)

Only valid for two-bracket linear inequality functions or quadratic functions.

Let us learn this with examples.

1. Solve for x, if $x^2 - 3x + 2 \le 0$

Bhaala method

$x^2 - 3x + 2 \le 0$
$(x - 1)(x - 2) \le 0$
⟵ (bhaala towards the function)
So, values of x will be within its critical points.
$\Rightarrow 1 \le x \le 2$
$\Rightarrow x \in [1, 2]$
Note: $\le$ or $<$ means ⟵ bhaala towards the function

Quick Query 1

(a) Solve for x, if $x^2 - 7x + 10 \geq 0$.

(b) Solve for x, if $x^2 - 6x + 8 < 0$.

Note

1.

$\sqrt{x^2} = |x|$

$\Rightarrow |x| = \sqrt{x^2}$

Similarly,

$|x + 1| = \sqrt{(x + 1)^2}$

$|x + 3| = \sqrt{(x + 3)^2}$

2.

Turning point is the value of x where the quantity inside the modulus is zero.

Turning point of $|x - 1|$ is $x = 1$

Turning point of $|x + 7|$ is $x = -7$

Turning point of $|x|$ is $x = 0$

3.

$0 \leq x^2 \leq 1$

We know that, negative value < positive value.

And we also know that square of a real number is always non-negative.

Therefore, we can ignore the left hand side of the inequality.

$\Rightarrow x^2 \leq 1$

Similarly, if $-1 \leq |x| \leq 1$

$\Rightarrow |x| \leq 1$ because $|x|$ is also always non-negative and hence will be greater than any negative value.

Ghoda method(for modulus function)

Example 1

Solve $|x| < 3$

Here, the critical point will be $x = 0$

Using the ghoda method,

Tie a horse at the critical point $x = 0$ with a 3 metre rope.

Here, the ghoda can eat the grass within the area that is three metres right of $x = 0$ and three metres left of $x = 0$

So, $x \in (-3, 3)$ or $-3 < x < 3$

Example 2

Solve |x| > 5

Here, the critical point is $x = 0$

Using the ghoda method,

Tie a horse at $x = 0$, with infinite length rope. However, the condition given is that the horse is not allowed to eat in the range of 5 metres, i.e., the horse can only eat the grass from the area that is more than 5 metres away from its critical point.

So here, $x \in (-\infty, -5) \cup (5, \infty)$

Note: < means within and > means beyond.

Solve for x, if |x| < 2.

Solution

Step 1:

We know that, $\sqrt{x^2} = |x|$

$\therefore \sqrt{x^2} < 2$

Since both sides are positive, squaring both sides, we get,

$x^2 < 4$

$\Rightarrow x^2 - 4 < 0$

$\Rightarrow (x + 2)(x - 2) < 0$ $\quad [\because a^2 - b^2 = (a + b)(a - b)]$

$\Rightarrow -2 < x < 2$

Solve for x, if |x| > 4.

Solution

Step 1:

We know that, $\sqrt{x^2} = |x|$

$\therefore \sqrt{x^2} > 4$

Since both sides are positive, squaring both sides, we get,

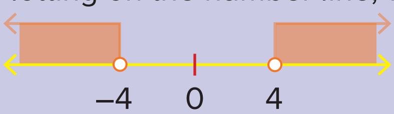
$x^2 > 16$

$\Rightarrow x^2 - 16 > 0$

$\Rightarrow (x + 4)(x - 4) > 0$ $\quad [\because a^2 - b^2 = (a + b)(a - b)]$

Step 2:

Plotting on the number line, we get,



$\Rightarrow x < -4$ or $x > 4$

$\therefore x \in (-\infty, -4) \cup (4, \infty)$

Solve for x, if $|x - 5| \le 2$.

Solution

Method 1

Step 1:

We know that, $\sqrt{x^2} = |x|$

$\therefore \sqrt{(x-5)^2} \le 2$

Since both sides are positive, squaring both sides, we get,

$(x-5)^2 - 4 \le 0$

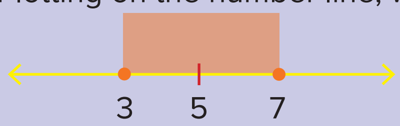
$\Rightarrow (x-5)^2 - 4 \le 0$

$\Rightarrow (x - 5 + 2)(x - 5 - 2) \le 0$ $[\because a^2 - b^2 = (a + b)(a - b)]$

$\Rightarrow (x-3)(x-7) \le 0$

Step 2:

Plotting on the number line, we get,



$\Rightarrow 3 \le x \le 7$

$\therefore x \in [3, 7]$

Method 2

$|x - 5| \le 2$

$-2 \le (x - 5) \le 2$

Add 5 in all the three expressions,

$-2 + 5 \le (x - 5) + 5 \le 2 + 5$

$3 \le x \le 7$

So, $x \in [3, 7]$

Quick Query 2

(a) Solve for x, if $|x| \ge 7$

(b) Solve for x , if $|x - 3| \le 6$

(c) Solve for x, if $\left|\frac{(3x + 1)}{3}\right| < 5$

Solve for x, if $|x - 1| \ge 7$.

Solution

Step 1:

We know that, $\sqrt{x^2} = |x|$

$\therefore \sqrt{(x-1)^2} \ge 7$

Since both sides are positive, squaring both sides, we get,

$(x - 1)^2 \geq 49$

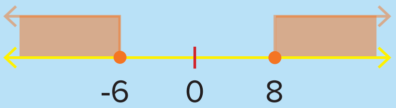
$\Rightarrow (x - 1)^2 - 49 \geq 0$

$\Rightarrow (x - 1 + 7)(x - 1 - 7) \geq 0$ $[\because a^2 - b^2 = (a + b)(a - b)]$

$\Rightarrow (x + 6)(x - 8) \geq 0$

Step 2:

Plotting on the number line, we get,



$\Rightarrow x \leq -6$ or $x \geq 8$

$\therefore x \in (-\infty, -6] \cup [8, \infty)$

Solve for x, if 2 < |x| < 6.

Solution

Step 1:

We have, $2 < |x| < 6$

Now, we can use the property of modulus,

$a < |x| < b \ \forall \ a, b > 0$

$\Rightarrow x \in (-b, -a) \cup (a, b)$

On comparing, we get,

$a = 2, b = 6$

$\Rightarrow x \in (-6, -2) \cup (2, 6)$

Solve for x, if -2 < |x| < 6.

Solution

Step 1:

We have, $-2 < |x| < 6$

Since the modulus always gives a non-negative value and positive value is always greater than negative value, we can rewrite the expression as

$0 \leq |x| < 6$

$\Rightarrow |x| < 6$

Step 2:

Now, we can use the property of modulus,

$|x| < a \ \forall \ a > 0$

$\Rightarrow -a < x < a$

On comparing, we get,

$a = 6$

$\Rightarrow -6 < x < 6$

$\Rightarrow x \in (-6, 6)$

Solve for x, if |x| ≤ 3 and |x| ≥ 1.

Solution

Step 1:

We have, $|x| \leq 3$ and $|x| \geq 1$

We will be solving these conditions as two separate cases.

Case 1: $|x| \leq 3$

We can use the property of modulus.

$|x| \leq a \;\forall\; a > 0$

$\Rightarrow -a \leq x \leq a$

On comparing, we get,

$a = 3$

$\Rightarrow -3 \leq x \leq 3$

Case 2: $|x| \geq 1$

We can use the property of modulus.

$|x| \geq a \;\forall\; a > 0$

$\Rightarrow x \leq -a$ or $x \geq a$

$\Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

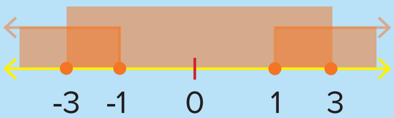
On comparing, we get,

$a = 1$

$\Rightarrow x \in (-\infty, -1] \cup [1, \infty)$

Step 2:

Since 'and' is mentioned between the conditions, we will have to take the intersection of the solutions we got in both the cases.



$\Rightarrow x \in [-3, -1] \cup [1, 3]$

Solve for x if $\frac{1 - |x|}{2 - |x|} \geq 0$.

Solution

Step 1:

We have,

$$\frac{1 - |x|}{2 - |x|} \geq 0$$

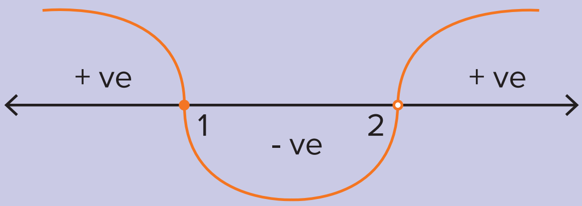
$$\Rightarrow \frac{|x| - 1}{|x| - 2} \geq 0$$

Now, let $|x| = t$

$$\Rightarrow \frac{t - 1}{t - 2} \geq 0$$

Step 2:

Now, we will use the wavy curve method.



From the curve, we can see that the expression is positive in $(-\infty, 1]$ and $(2, \infty)$. We did not include 2 because at $t = 2$ the expression $\frac{t-1}{t-2}$ becomes undefined.

$\Rightarrow t \leq 1 \cup t > 2$

On re-substituting $t = |x|$, we get,

$|x| \leq 1 \cup |x| > 2$

Step 3:

We will be solving these conditions as two separate cases.

Case 1: $|x| \leq 1$

We can use the property of modulus.

$|x| \leq a \ \forall \ a > 0$

$\Rightarrow -a \leq x \leq a$

On comparing, we get,

$a = 1$

$\Rightarrow -1 \leq x \leq 1$

Case 2: $|x| > 2$

We can use the property of modulus.

$|x| > a \ \forall \ a > 0$

$\Rightarrow x < -a$ or $x > a$

$\Rightarrow x \in (-\infty, -a) \cup (a, \infty)$

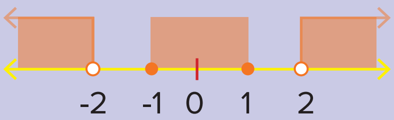
On comparing, we get,

$a = 2$

$\Rightarrow x \in (-\infty, -2) \cup (2, \infty)$

Step 4:

Now, plotting the two sets of solutions on the number line, we get,



We have to take the union of all the sets of solutions.

$\therefore x \in (-\infty, -2) \cup [-1, 1] \cup (2, \infty)$

Property 4

If $x, y \in \mathbb{R}$

$\Rightarrow |x + y| \leq |x| + |y|$

Case 1: $|x + y| = |x| + |y|$

This holds true when $xy \geq 0$

In other words, when both x, y are of the same sign or atleast one of them is 0

Case 2: $|x + y| < |x| + |y|$

This holds true when x, y are of the opposite sign i.e. $xy < 0$.

Find the solution set of $|x - 1| + |2x - 3| = |3x - 4|$.

a. $(-\infty, 1) \cup \left[\frac{3}{2}, \infty\right)$ b. $(-\infty, 1] \cup \left[\frac{3}{2}, \infty\right)$ c. $(-\infty, 1] \cup \left(\frac{3}{2}, \infty\right)$ d. Ø

Solution

Step 1:

We have,

$|x - 1| + |2x - 3| = |3x - 4|$

On comparing it with $|a| + |b| = |a + b|$, we get,

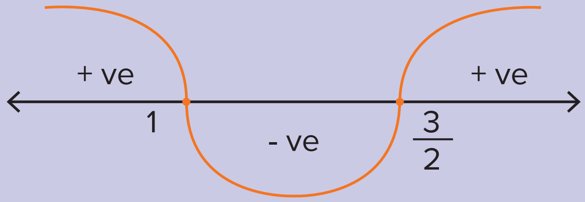
$a = x - 1, b = 2x - 3$

Now, for the equality to hold, $ab \geq 0$

$\Rightarrow (x - 1)(2x - 3) \geq 0$

Step 2:

Using the wavy curve method,



We can observe that the inequality holds true for $-\infty < x \leq 1$ or $\frac{3}{2} \leq x < \infty$

$\Rightarrow x \in (-\infty, 1] \cup [\frac{3}{2}, \infty)$

Therefore, option b is the correct answer.

Solve for x, if |4x + 1| - |3x - 2| = |x + 3|.

Solution

Step 1:

We have, $|4x + 1| - |3x - 2| = |x + 3|$

$\Rightarrow |4x + 1| = |3x - 2| + |x + 3|$

On comparing with $|a| + |b| = |a + b|$, we get,

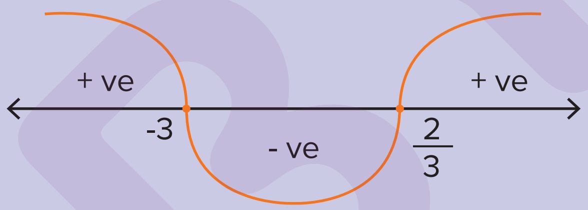
$a = 3x - 2$, $b = x + 3$

Now, for the equality to hold, $ab \geq 0$

$\Rightarrow (3x - 2)(x + 3) \geq 0$

Step 2:

Using the wavy curve method,



We can observe that the inequality holds true for $-\infty < x \leq -3$ or $\frac{2}{3} \leq x < \infty$

$\Rightarrow x \in (-\infty, -3] \cup [\frac{2}{3}, \infty)$

Find the total number of integral solution(s) of $\left|\frac{3x}{2} - 1\right| + \left|2 - \frac{x}{2}\right| = |x + 1|$

a. 1 **b.** 2 **c.** 3 **d.** 4

Solution

Step 1:

We have, $\left|\frac{3x}{2} - 1\right| + \left|2 - \frac{x}{2}\right| = |x + 1|$

$$\Rightarrow \left|\frac{3x}{2} - 1\right| + \left|2 - \frac{x}{2}\right| = |x + 1| = \left|\left(\frac{3x}{2} - 1 + 2 - \frac{x}{2}\right)\right|$$

Step 2:

On comparing with |a| + |b| = |a + b|, we get,

$a = \frac{3x}{2} - 1, b = 2 - \frac{x}{2}$

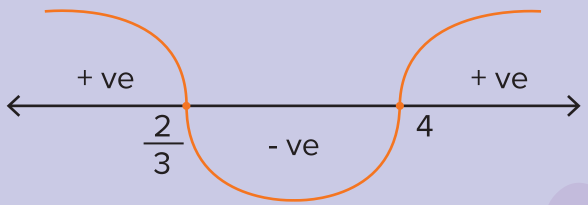
Now, for the equality to hold, $ab \geq 0$

$$\Rightarrow \left(\frac{3x}{2} - 1\right)\left(2 - \frac{x}{2}\right) \geq 0$$

$$\Rightarrow \left(\frac{3x}{2} - 1\right)\left(\frac{x}{2} - 2\right) \leq 0$$

Step 3:

Using the wavy curve method,



We can observe that the inequality holds true for $\frac{2}{3} \leq x \leq 4$

$\Rightarrow x \in [\frac{2}{3}, 4]$

The integers lying in this interval are 1, 2, 3 and 4.

So, option d) is the correct answer.

Property 5

If $x, y \in \mathbb{R}$

$\Rightarrow |x - y| \leq |x| + |y|$

Case 1: $|x - y| = |x| + |y|$

This holds true when $xy \leq 0$

In other words, when x, y are of the opposite sign or atleast one of them is 0.

Case 2: $|x - y| < |x| + |y|$

This holds true when both x,y are of the same sign i.e. $xy > 0$.

If the solution set of $|7x - 5| + |6x - 11| = |x + 6|$ is [p, q], then find (p + q).

a. $\frac{47}{42}$ b. $\frac{97}{42}$ c. $\frac{107}{42}$ d. $\frac{77}{42}$

Solution

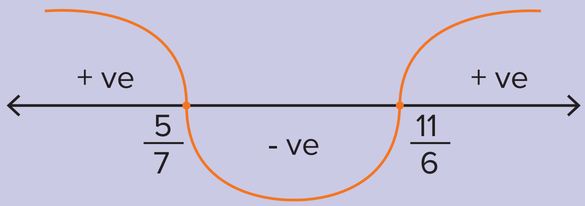
On comparing with |a| + |b| = |a - b|

a = (7x - 5) and b = (6x - 11)

Also, $ab \leq 0$

Therefore, $(7x - 5)(6x - 11) \leq 0$

Using the wavy curve method, we get,



$x \in \left[\frac{5}{7}, \frac{11}{6}\right]$, $p = \frac{5}{7}$ and $q = \frac{11}{6}$, $p + q = \frac{107}{42}$

Concept Check

1) Solve for x, if $|x + 3| \leq 4$.

2) Solve for x, if $1 < |x - 1| < 4$.

3) Find the smallest value of $x \in N$, satisfying the equation $\left|\frac{x}{(1-x)}\right| + |x| = \left|\frac{x^2}{(1-x)}\right|$.

(a) 4 (b)3 (c)2 (d)1

Summary sheet

Key Takeaways

$|x| \geq 0 \ \forall \ x \in R$

$|x| = 0 \Leftrightarrow x = 0$

$|x| = a \Leftrightarrow x = \pm a$, where $a > 0$

$\sqrt{x^2} = |x| = \pm x \ \forall \ x \in R$

$|x| = |-x| \ \forall \ x \in R$

$|x| = |y| \Rightarrow x = \pm y \ \forall \ x, y \in R$

$|x y| = |x| \, |y| \ \forall \ x, y \in R$

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \ \forall \ x, y \in R$, and $y \neq 0$

Key properties for modulus inequalities

$|x + y| \leq |x| + |y| \ \forall \ x, y \in R$

$|x - y| \leq |x| + |y| \ \forall \ x, y \in R$

$|x| \leq a$, where $a > 0 \Leftrightarrow -a \leq x \leq a$ or $x \in [-a, a]$

$|x| < a$, where $a > 0 \Leftrightarrow -a < x < a$ or $x \in (-a, a)$

$|x| \geq a$, where $a > 0 \Leftrightarrow x \leq -a$ or $x \geq a \Leftrightarrow x \in (-\infty, -a] \cup [a, \infty)$

$|x| > a$, where $a > 0 \Leftrightarrow x < -a$ or $x > a \Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$

$a \leq |x| \leq b$, where $a, b > 0 \Leftrightarrow x \in [-b, -a] \cup [a, b]$

$a < |x| < b$, where $a, b > 0 \Leftrightarrow x \in (-b, -a) \cup (a, b)$

Mind Map

Modulus Properties

- $|xy| = |x|\,|y|\ \forall\ x, y \in R$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}\ \forall\ x, y \in R$, and $y \neq 0$
- $|x| = |y| \Rightarrow x = \pm y\ \forall\ x, y \in R$

Modulus Inequality

- $|x + y| \leq |x| + |y|\ \forall\ x, y \in R$
- $|x - y| \leq |x| + |y|\ \forall\ x, y \in R$
- $|x| \leq a$, where $a > 0 \Leftrightarrow x \in [-a, a]$
- $|x| < a$, where $a > 0 \Leftrightarrow -a < x < a$ or $x \in (-a, a)$
- $|x| \geq a$, where $a > 0 \Leftrightarrow x \leq -a$ or $x \geq a \Leftrightarrow x \in (-\infty, -a] \cup [a, \infty)$
- $|x| > a$, where $a > 0 \Leftrightarrow x < -a$ or $x > a \Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$
- $a \leq |x| \leq b$, where $a, b > 0 \Leftrightarrow x \in [-b, -a] \cup [a, b]$
- $a < |x| < b$, where $a, b > 0 \Leftrightarrow x \in (-b, -a) \cup (a, b)$

Self-Assessment

1. Solve for x if $\frac{|2|}{(x - 4)} > 1$.
2. Solve for x if, $1 < |x - 3| < 4$.

Answers

Quick Query

1) a)

$x^2 - 7x + 10 \geq 0$

$(x - 2)(x - 5) \geq 0$

$x \in (-\infty, 2] \cup [5, \infty)$

b)

$x^2 - 6x + 8 < 0$

$(x - 2)(x - 4) < 0$

$x \in (2, 4)$

2) a)

Step 1: We know that, $\sqrt{x^2} = |x|$

$\sqrt{x^2} \geq 7$

Since both sides are positive we can take square on both sides.

On squaring both sides, we get,

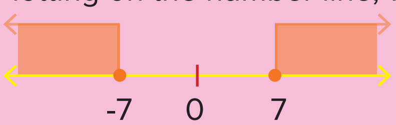
$x^2 \geq 49$

$\Rightarrow x^2 - 49 \geq 0$

$\Rightarrow (x + 7)(x - 7) \geq 0$ $\quad [\because a^2 - b^2 = (a + b)(a - b)]$

Step 2:

Plotting on the number line, we get,



$\Rightarrow x \leq -7$ or $x \geq 7$

$\therefore x \in (-\infty, -7] \cup [7, \infty)$

b) $|x - 3| \leq 6$

$-6 \leq (x - 3) \leq 6$

Adding 3,

$-3 \leq x \leq 9$

$x \in [-3, 9]$

c) $\left|\frac{(3x + 1)}{3}\right| < 5$

$\left|x + \frac{1}{3}\right| < 5$

$-5 < \left(x + \frac{1}{3}\right) < 5$;

Subtract 1/3 from the above expression

$\frac{-16}{3} < x < \frac{14}{3}$

$x \in \left(\frac{-16}{3}, \frac{14}{3}\right)$

Concept Check

1)

Step 1:

We know that,

$\sqrt{x^2} = |x|$

$\therefore \sqrt{(x + 3)^2} \leq 4$

Since both sides are positive, squaring both sides, we get,

$(x + 3)^2 \leq 16$

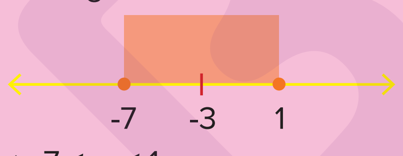
$\Rightarrow (x + 3)^2 - 16 \leq 0$

$\Rightarrow (x + 3 + 4)(x + 3 - 4) \leq 0$ $[\because a^2 - b^2 = (a + b)(a - b)]$

$\Rightarrow (x + 7)(x - 1) \leq 0$

Step 2:

Plotting on the number line, we get,,



$\Rightarrow -7 \leq x \leq 1$

$\therefore x \in [-7, 1]$

2) Step 1:

We have, $1 < |x - 1| < 4$

Now, we can use the property of modulus,

$a < |x| < b \ \forall \ a, b > 0$

$\Rightarrow x \in (-b, -a) \cup (a, b)$

On comparing, we get,

$a = 1, b = 4$

$\Rightarrow -4 < x - 1 < -1$ or $1 < x - 1 < 4$

Adding 1 to the inequality, we get,

$\Rightarrow -3 < x < 0$ or $2 < x < 5$

$\Rightarrow x \in (-3, 0) \cup (2, 5)$

3)

Step 1:

We have,

$$\left|\frac{x}{1-x}\right| + |x| = \left|\frac{x^2}{1-x}\right|$$

Now,

$$\left|\frac{x^2}{1-x}\right| = \left|\frac{x}{1-x} - x\right|$$

$$\Rightarrow \left|\frac{x}{1-x}\right| + |x| = \left|\frac{x}{1-x} - x\right|$$

Step 2:

On comparing with $|a - b| = |a| + |b|$, we get,

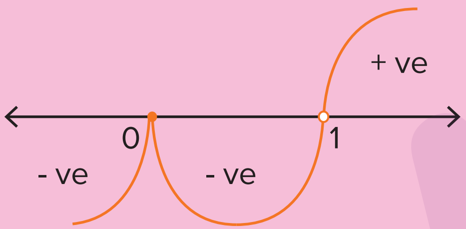
$a = \frac{x}{1-x}$, $b = x$

Now, for the equality to hold, $ab \le 0$

$$\Rightarrow \frac{x}{1-x} * x \le 0 \Rightarrow \frac{x^2}{x-1} \ge 0$$

Step 3:

On using the wavy curve method, we get,



We can observe that the inequality holds true for $1 < x < \infty$. Note that 1 is not included because at $x = 1$, the expression becomes undefined. Also, observe that $x = 0$ satisfies the given equation.

$\Rightarrow x \in (1, \infty) \cup \{0\}$

The smallest natural number in this region is 2. Therefore, option c is the correct answer.

Self-Assessment

1)

Step 1:

We have,

$$\left|\frac{2}{x-4}\right| > 1$$

$$\Rightarrow \frac{|2|}{|x-4|} > 1$$

$$\Rightarrow \frac{2}{|x-4|} > 1$$

Step 2:

We know that, $|x - 4| > 0$ (for $x \ne 4$)

Therefore, we can cross multiply it.

$2 > |x - 4|$

$\Rightarrow |x - 4| < 2$

Step 3:

Now, we can use the property of modulus,

$| x | < a \ \forall \ a > 0$

$\Rightarrow -a < x < a$

On comparing, we get,

$a = 2$

$\Rightarrow -2 < x - 4 < 2$

Adding 4 to the inequality, we get,

$-2 + 4 < x < 2 + 4$

$\Rightarrow 2 < x < 6$

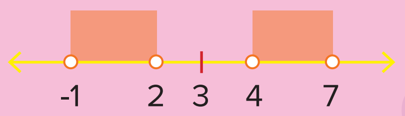
Observe that at x = 4, the expression $\frac{2}{|x - 4|}$ becomes undefined. So we will exclude that value.

$\Rightarrow x \in (2, 6) - \{4\}$

2) Given, $1 < |x - 3| < 4$

Turning point of $|x - 3|$ is $x = 3$,

Therefore,



$x \in (-1, 2) \cup (4, 7)$