MATHEMATICS

RELATIONS AND FUNCTIONS

SOLVING PROBLEMS ON FINDING DOMAIN OF FUNCTIONS

NOTES

What you already know

- Definition of a function
- Domain and range of a function
- Types of functions

What you will learn

- Finding the domain of various functions

Domain

To find domain of basic types of functions:

If $y = \frac{1}{f(x)}$ then $f(x) \neq 0$

If $y = \sqrt{f(x)}$ then $f(x) \geq 0$

If $y = \frac{1}{\sqrt{f(x)}}$ then $f(x) > 0$

Method to find the domain if addition or subtraction of two or more functions are given,

$$h(x) = f(x) \pm g(x)$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$D = D_1 \cap D_2$$

where D, D_1 and D_2 are domains of $h(x), f(x)$ and $g(x)$.

Find the domain of the following functions:

Solution

1. $f(x) = \sqrt{x^2 - x - 20} + \frac{1}{\sqrt{x^2 - 5x - 14}}$

Case 1: $x^2 - x - 20 \geq 0$

$\Rightarrow (x - 5)(x + 4) \geq 0$

$\Rightarrow x \leq -4 \cup x \geq 5$

$\Rightarrow x \in (-\infty, -4] \cup [5, \infty)$

Case 2: $x^2 - 5x - 14 > 0$

$\Rightarrow (x - 7)(x + 2) > 0$

$\Rightarrow x < -2 \cup x > 7$

$\Rightarrow x \in (-\infty, -2) \cup (7, \infty)$

By taking intersection of Case 1 and Case 2, we get domain as follows:

$x \in (-\infty, -4] \cup (7, \infty)$

2. $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

Analytical method

$|x| - x > 0$

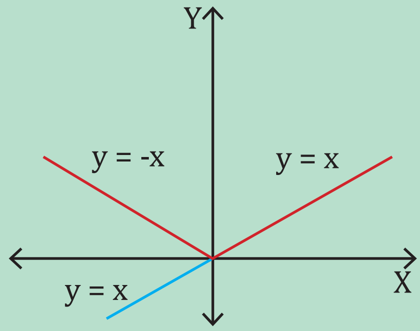
When $x \geq 0$,
$\Rightarrow x > x$
Not possible

When $x < 0$,
$\Rightarrow -x > x$
$\Rightarrow 2x < 0$
$\Rightarrow x < 0$

$\Rightarrow x \in (-\infty, 0)$

Graphical method

$|x| > x$



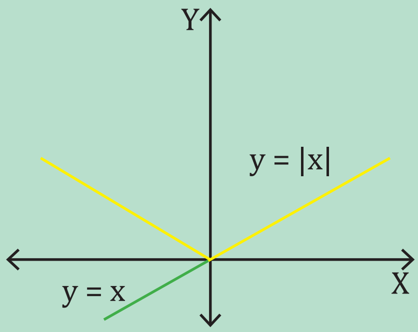
$\Rightarrow x \in (-\infty, 0)$

Hit and trial

$|x| > x$
Let $x = 3, 3 > 3$
Not possible
Let $x = -3, 3 > -3$
True
Let $x = 3.1, 3.1 > 3.1$
Not possible
Let $x = -3.1, 3.1 > -3.1$
We can see that the inequality holds for negative values of x
$\Rightarrow x \in (-\infty, 0)$

3. $f(x) = \dfrac{1}{\sqrt{x - |x|}}$

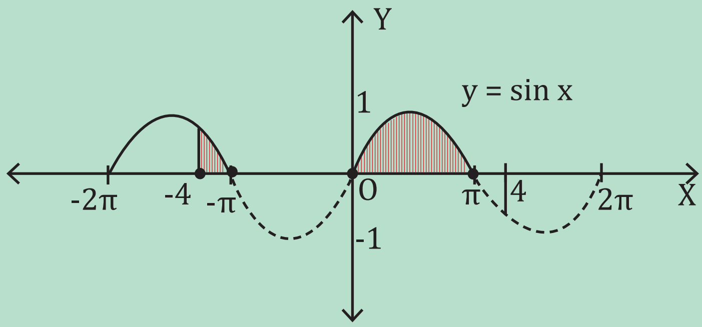
$x - |x| > 0$
$\Rightarrow |x| < x$
From the graph, we can see that the green line does not exceed the yellow line at any point.
$x = \Phi$



4. $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

sin $x \geq 0$ is possible if $y = \sin x$, is above the X-axis.

$16 - x^2 \geq 0$
$\Rightarrow x^2 - 16 \leq 0$
$\Rightarrow (x - 4)(x + 4) \leq 0$
$\Rightarrow -4 \leq x \leq 4$
By taking intersection of above two sets of x we get domain as
$x \in [-4, -\pi] \cup [0, \pi]$



Note

For the composite function $f(g(x))$ we have the rule,
Domain of $f(g(x))$ = Domain of $g(x)$, if Domain of function f is $\mathbb{R}$.

Find the domain of the following functions:

Solution

1. $f(x) = \sin\left(\sqrt{x-1} + \sqrt{6-x}\right)$

$f(x) = \sin(g(x));\ g(x) = \sqrt{x-1} + \sqrt{6-x}$

$\Rightarrow$ Domain of $f(x) = \mathbb{R}$

We know, Domain of $f(g(x))$ = Domain of $g(x)$

Domain of $\sqrt{x-1} + \sqrt{6-x}$,

$x - 1 \geq 0 \Rightarrow x \geq 1$

$6 - x \geq 0 \Rightarrow x \leq 6$

By taking intersection of above two sets of x we get domain as $x \in [1, 6]$

2. $f(x) = 4^{-\left(\sqrt{x-1} + \sqrt{6-x}\right)}$

$f(x) = 4^{-g(x)}$;

$g(x) = \sqrt{x-1} + \sqrt{6-x}$

Domain of $\sqrt{x-1} + \sqrt{6-x}$,

$x - 1 \geq 0 \Rightarrow x \geq 1$

$6 - x \geq 0 \Rightarrow x \leq 6$

By taking intersection of above two sets of x we get domain as $x \in [1, 6]$

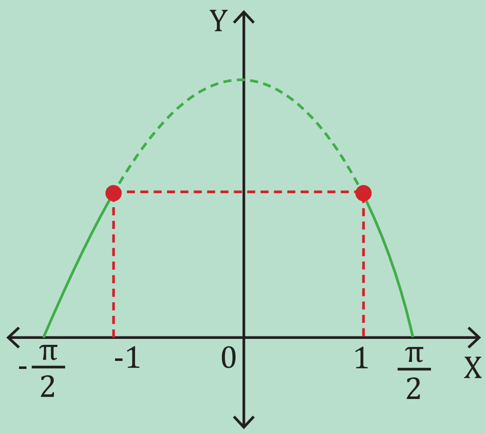
3. $f(x) = \sqrt{\cos(\sin x)}$

For function to exist, $\cos(\sin x) \geq 0$

Step 1:

$f(x) = \sqrt{\cos\theta};\ \theta = \sin x$

$\Rightarrow \cos\theta \geq 0$ is that part of $\cos\theta$ which is above X-axis.



Step 2:

Also, $\theta = \sin x \in [-1, 1]$;

i.e, θ is bounded to $[-1, 1]$.

Step 3:

Part of $\cos\theta$ in $[-1, 1]$ is already above X-axis.

Hence, $x \in \mathbb{R}$.

Note

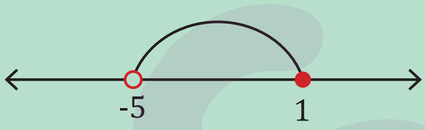
1. Domain of $(f(x))^{\frac{1}{4}}, (f(x))^{\frac{1}{6}}, \ldots, (f(x))^{\frac{1}{\text{Even}}}$ = Domain of $\sqrt{f(x)}$

2. Domain of $(f(x))^{\frac{1}{3}}, (f(x))^{\frac{1}{5}}, \ldots, (f(x))^{\frac{1}{\text{Odd}}}$ = Domain of $f(x)$

Find the domain of the following function: $f(x) = \left(\frac{1-x}{5+x}\right)^{\frac{1}{12}}$

Solution

$$f(x) = \left(\frac{1-x}{5+x}\right)^{\frac{1}{12}}$$



Domain of $(f(x))^{\frac{1}{\text{Even}}}$ = Domain of $\sqrt{f(x)}$

$\Rightarrow$ Domain of $\sqrt{\frac{1-x}{5+x}} \Rightarrow \frac{1-x}{5+x} \geq 0$

$\Rightarrow x \in (-5, 1]$

Logarithmic Functions

Find the domain of the following functions:

1. $\log_3\left(\log_{\frac{1}{3}}\left(x^2 + 10x + 5\right)\right)$ **2.** $f(x) = \log_{100x}\left(\frac{2\log_{10} x + 1}{-x}\right)$

Solution

1. $\log_3\left(\log_{\frac{1}{3}}\left(x^2 + 10x + 5\right)\right)$

Step 1:

For $\log_3(g(x))$,

Base = 3 > 0, Also, $3 \neq 1$

Step 2:

Now, $\log_{\frac{1}{3}}\left(x^2 + 10x + 5\right) > 0$

$\Rightarrow x^2 + 10x + 5 < 1 \Rightarrow x^2 + 10x + 4 < 0$

For $x^2 + 10x + 4 = 0$,

$\Rightarrow x = \frac{-10 \pm \sqrt{100 - 16}}{2}$

$\Rightarrow x = \frac{-10 \pm 2\sqrt{21}}{2} \Rightarrow x = -5 \pm \sqrt{21}$

For $x^2 + 10x + 4 < 0$,

$x \in \left(-5 - \sqrt{21}, -5 + \sqrt{21}\right)$

Step 3:

For $\log_{\frac{1}{3}}(h(x))$

Base $= \frac{1}{3} > 0$, Also, $\frac{1}{3} \neq 1$

Step 4:

Now, for $x^2 + 10x + 5 = 0$,

$\Rightarrow x = \frac{-10 \pm \sqrt{100 - 20}}{2}$

$\Rightarrow x = \frac{-10 \pm 2\sqrt{20}}{2}$

$\Rightarrow x = -5 \pm \sqrt{20}$

So, for $x^2 + 10x + 5 > 0$

$x \in \mathbb{R} - \left(-5 - \sqrt{20}, -5 + \sqrt{20}\right)$

Step 5:

$x \in \left(-5 - \sqrt{21}, -5 - \sqrt{20}\right) \cup \left(-5 + \sqrt{20}, -5 + \sqrt{21}\right)$

2. $f(x) = \log_{100x}\left(\dfrac{2\log_{10} x + 1}{-x}\right)$

For function to exist,

a. Base = $100x > 0$; $x > 0$, Also $100x \neq 1$; $x \neq \dfrac{1}{100}$

b. $\dfrac{2\log_{10} x + 1}{-x} > 0 \Rightarrow \dfrac{2\log_{10} x + 1}{x} < 0$

$\Rightarrow 2\log_{10} x + 1 < 0 \Rightarrow 2\log_{10} x < -1$

$\Rightarrow \log_{10} x < -\dfrac{1}{2} \Rightarrow x < 10^{-\frac{1}{2}} \Rightarrow x < \dfrac{1}{\sqrt{10}}$

c. In $2\log_{10} x + 1, x > 0$

Considering (a), (b) and (c),

$x \in \left(0, \dfrac{1}{100}\right) \cup \left(\dfrac{1}{100}, \dfrac{1}{\sqrt{10}}\right)$

Find the domain of the following functions:

1. $\log_{10}|1 + x^3|$ **2.** $f(x) = \log_{10}|4x - 1|$ **3.** $\log_{10}\left[1 - \log_{10}\left(x^2 - 5x + 16\right)\right]$

Solution

1. $\log_{10}|1 + x^3|$

For function to exist,

a. Base $= 10 \neq 0$, also $10 > 1$

b. $1 + x^3 \neq 0$

$\Rightarrow x^3 \neq -1 \Rightarrow x \neq -1$

$\Rightarrow x \in \mathbb{R} - \{-1\}$

2. $f(x) = \log_{10}|4x - 1|$

For function to exist,

a. Base $= 10 \neq 0$, also $10 > 1$

b. $4x - 1 \neq 0 \Rightarrow x \neq \dfrac{1}{4}$

$\Rightarrow x \in \mathbb{R} - \left\{\dfrac{1}{4}\right\}$

3. $\log_{10}\left[1 - \log_{10}\left(x^2 - 5x + 16\right)\right]$

For function to exist,

a. Base $= 10 \neq 0$, also, $10 > 1$

b. $1 - \log_{10}\left(x^2 - 5x + 16\right) > 0$

$\Rightarrow \log_{10}\left(x^2 - 5x + 16\right) < 1 \Rightarrow x^2 - 5x + 16 < 10$

$\Rightarrow x^2 - 5x + 6 < 0 \Rightarrow (x - 2)(x - 3) < 0 \Rightarrow 2 < x < 3$

c. $x^2 - 5x + 16 > 0$

$\Rightarrow D = 25 - 4 \times 16 = -ve \Rightarrow x \in \mathbb{R}$

$\Rightarrow x \in (2, 3)$

Polynomial-Based Questions (Type 1)

If f(x) is a polynomial with degree 5 with leading coefficient 2 and f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 4, and f(5) = 5, then find f(6) .

Solution

Step 1:

Let $g(x) = f(x) - x$

So, $g(1) = f(1) - 1 = 1 - 1 = 0$

$g(2) = f(2) - 2 = 2 - 2 = 0$

$g(3) = f(3) - 3 = 3 - 3 = 0$

$g(4) = f(4) - 4 = 4 - 4 = 0$

$g(5) = f(5) - 5 = 5 - 5 = 0$

Step 2:

From the step 1 calculations, we can conclude that,

1. $g(x)$ is a polynomial having roots 1, 2, 3, 4, and 5.
2. $g(x)$ is also a 5 degree polynomial.

Step 3:

Now, $g(x) = (x-1)(x-2)(x-3)(x-4)(x-5)$

$f(x) - x = 2(x-1)(x-2)(x-3)(x-4)(x-5)$ ($\because$ the leading coefficient is 2.)

$\Rightarrow f(x) = 2(x-1)(x-2)(x-3)(x-4)(x-5) + x$

$\Rightarrow f(6) = 2(6-1)(6-2)(6-3)(6-4)(6-5) + 6 = 246$

Concept Check

Find the domain of the following functions.

1. $f(x) = \dfrac{\sqrt{x+4}}{\ln(x-8)}$
2. $f(x) = \dfrac{1}{\sqrt{x^2 - 4x}}$
3. $f(x) = \sqrt{\cos 2x} + \sqrt{4 - x^2}$
4. $f(x) = \sqrt{\dfrac{1 - 5^x}{7^{-x} - 7}}$
5. $f(x) = \left(\dfrac{1-x}{5+x}\right)^{\frac{1}{11}}$
6. $f(x) = \sqrt{\log_3 \cos(\sin x)}$
7. $f(x) = \log_{10}(\log_e x)$
8. If f(x) is a polynomial with degree 4 with leading coefficient 3 and f(1) = 2, f(2) = 5, f(3) = 10, and f(5) = 26, then find f(4) .

Summary Sheet

Key takeaways

- To find the domain of basic types of functions:

  If $y = \dfrac{1}{f(x)}$ then $f(x) \neq 0$

  If $y = \sqrt{f(x)}$ then $f(x) \geq 0$

  If $y = \dfrac{1}{\sqrt{f(x)}}$ then $f(x) > 0$

- When we need to find the domain of two or more functions,

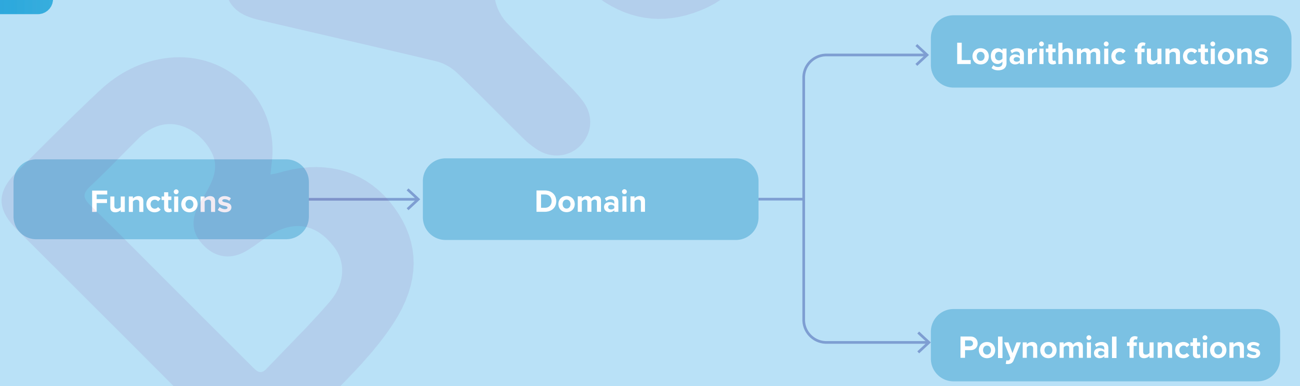
  $$h(x) = f(x) \pm g(x)$$
  $$\downarrow \quad \downarrow \quad \downarrow$$
  $$D = D_1 \cap D_2$$

  where D, D_1 and D_2 are domains of $h(x), f(x)$ and $g(x)$.

- 1. Domain of $(f(x))^{\frac{1}{4}}, (f(x))^{\frac{1}{6}}, \ldots, (f(x))^{\frac{1}{\text{Even}}}$ = Domain of $\sqrt{f(x)}$

  2. Domain of $(f(x))^{\frac{1}{3}}, (f(x))^{\frac{1}{5}}, \ldots, (f(x))^{\frac{1}{\text{Odd}}}$ = Domain of $f(x)$

Mind Map



Self-Assessment

Find the domain of the function: $f(x) = \left(\dfrac{x-1}{x-2}\right)^{3x+4}$

Answers

Concept Check

1.

$f(x) = \sqrt{x+4} \times \frac{1}{\ln(8-x)}$

$= \sqrt{x+4} \times \log_{8-x}(e)$

We know the base of the logarithmic function cannot be negative or 1.

$x + 4 \geq 0 \qquad 8 - x \neq 1 \qquad 8 - x > 0$

$\Rightarrow x \geq -4 \qquad \Rightarrow x \neq 7 \qquad \Rightarrow x < 8$

By taking intersection of above three sets of x we get domain as $x \in [-4, 8) - \{7\}$

2.

$x^2 - 4x > 0$

$\Rightarrow x(x-4) > 0$

$\Rightarrow x < 0 \cup x > 4$

$\Rightarrow x \in (-\infty, 0) \cup (4, \infty)$

$\Rightarrow x \in \mathbb{R} - [0, 4]$

3.

$\cos 2x \geq 0$

Is possible if $y = \cos 2x$, is above the X-axis.

$4 - x^2 \geq 0$

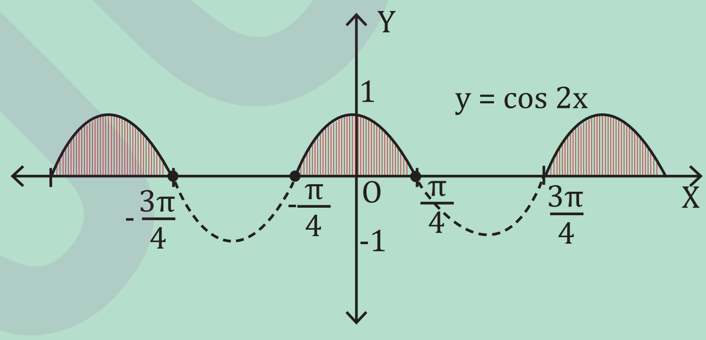
$\Rightarrow x^2 - 4 \leq 0$

$\Rightarrow (x-2)(x+2) \leq 0$

$\Rightarrow -2 \leq x \leq 2$

By taking intersection of above two sets of x we get domain as

$x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

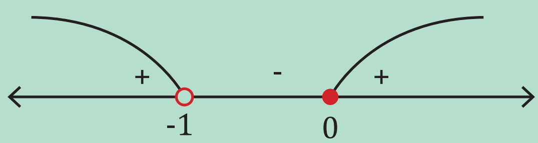


4.

$\frac{1 - 5^x}{7^{-x} - 7} \geq 0$

Multiplying numerator and denominator by 7^x

$\frac{(5^x - 1)7^x}{7^{x+1} - 1} \geq 0$



Now finding the critical points,

$5^x - 1 = 0$

$\Rightarrow 5^x = 5^0$

$\Rightarrow x = 0$

$7^{x+1} - 1 \neq 0$

$\Rightarrow 7^{x+1} \neq 7^0$

$\Rightarrow x \neq -1$

$\Rightarrow x \geq 0$ and $x < -1$

On taking intersection of above, we get domain as

$x \in (-\infty, -1) \cup [0, \infty)$

5.

$$f(x) = \left(\frac{1-x}{5+x}\right)^{\frac{1}{11}}$$

$\Rightarrow$ Domain of $(f(x))^{\frac{1}{Odd}}$ = Domain of $f(x)$

$\Rightarrow$ Domain of $\frac{1-x}{5+x} \Rightarrow 5+x \neq 0$

$\Rightarrow x \neq -5$

$\Rightarrow x \in \mathbb{R} - \{-5\}$

6.

$f(x) = \sqrt{\log_3 \cos(\sin x)}$; Now, for the function to exist,

a. Base $= 3 > 0$, also $3 \neq 1$

b. $\log_3 \cos(\sin x) \geq 0$

$\Rightarrow \cos(\sin x) \geq 1$ i,e $\cos\theta \geq 1$

But $\cos\theta = 1$ is the only possible solution.

$\Rightarrow \cos(\sin x) = 1$

$\Rightarrow \sin x = 0$

$\Rightarrow x = n\pi, n \in \mathbb{Z}$

7.

$f(x) = \log_{10}|\log_e x|$

For the function to exist,

a. Base $= 10 \neq 0$, also, $10 > 1$

b. $\log_e x \neq 0 \Rightarrow x \neq 1$. Also, $x > 0$

c. Base $= e \neq 0$, also $e > 1$

$\Rightarrow x \in (0, \infty) - \{1\}$

8.

Step 1:

Let $g(x) = f(x) - (x^2 + 1)$

So, $g(1) = f(1) - (1+1) = 2 - 2 = 0$

$g(2) = f(2) - (4+1) = 5 - 5 = 0$

$g(3) = f(3) - (9+1) = 10 - 10 = 0$

$g(5) = f(5) - (25+1) = 26 - 26 = 0$

Step 2:

From the calculations, we can conclude that,

1. $g(x)$ is a polynomial having roots 1, 2, 3, 5.
2. $g(x)$ is also a 4 degree polynomial.

Step 3:

Now, $g(x) = (x-1)(x-2)(x-3)(x-5)$

$f(x) - (x^2+1) = 3(x-1)(x-2)(x-3)(x-5)$ ($\because$ The leading coefficient is 3.)

$\Rightarrow f(x) = 3(x-1)(x-2)(x-3)(x-5) + (x^2+1)$

$\Rightarrow f(4) = 3(4-1)(4-2)(4-3)(4-5) + 17 = 17 - 18 = -1$

Self-Assessment

As $\left(\frac{x-1}{x-2}\right)^{3x+4}$ is in the form of $h(x)^{g(x)}$.

Here,

$g(x) = 3x + 4$ is defined for $\mathbb{R}$.....(1)

$h(x) = \frac{x-1}{x-2} > 0$

$\Rightarrow \frac{(x-1)(x-2)}{(x-2)^2} > 0$

$\Rightarrow (x-1)(x-2) > 0$

By using the wavy curve method, we get the following:

$x \in (-\infty, 1) \cup (2, \infty)$........(2)

On taking the intersection of (1) and (2), we get,

$x \in (-\infty, 1) \cup (2, \infty)$