MATHEMATICS

RELATIONS AND FUNCTIONS

INTRODUCTION TO FUNCTIONS

NOTES

What you already know

- Set theory
- Cartesian product
- Relations

What you will learn

- Function
- Domain, co-domain and range

Show that relation R defined on the set of real numbers such that $R = \{(a, b) : a > b\}$ is transitive.

Solution

Let $(a, b) \in R$ and $(b, c) \in R$.
$\Rightarrow a > b$ and $b > c$
$\Rightarrow a > c$
$\Rightarrow (a, c) \in R$
Therefore, R is a transitive relation.

Let T be the set of all the triangles in a plane with R as a relation given by $R = \{(T_1, T_2)$: T_1 is congruent to $T_2\}$. Show that R is an equivalence relation.

Solution

(1) Every triangle is congruent to itself.
Therefore, R is reflexive.

(2) $(T_1, T_2) \in R : T_1$ is congruent to T_2.
$\Rightarrow T_2$ is congruent to T_1.
Therefore, R is symmetric.

(3) $(T_1, T_2) \in R : T_1$ is congruent to T_2.
$(T_2, T_3) \in R : T_2$ is congruent to T_3.
$\Rightarrow T_1$ is congruent to T_3.
Therefore, R is transitive.
Thus, the given relation R is an equivalence relation.

Functions

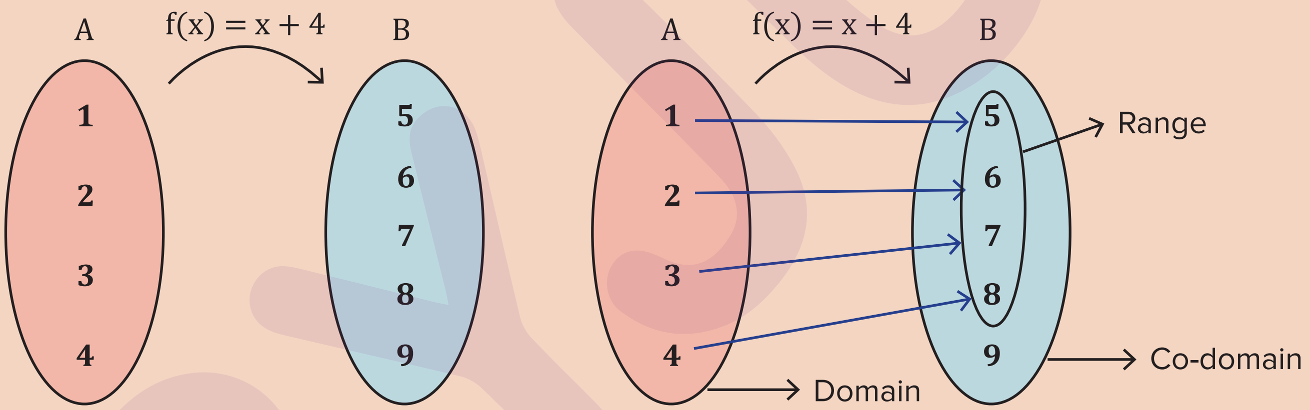
If A and B are two non-empty sets such that each element of A is associated with a unique element of B under the rule $f: A \rightarrow B$ (where A is the domain and B is the codomain of the function), then $f: A \rightarrow B; y = f(x)$ is known as the function.
Here, y is a dependent variable and x is an independent variable.

Note

- A function is also known as mapping, association, or correspondance.
- Range $\subseteq$ Codomain

Example:

If $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9\}$, then represent $f(x) = x + 4$ using mapping.



Here, Range $\subseteq$ Codomain
Also, $f(1) = 5$
Here, 5 is the image of 1 and 1 is the preimage of 5.
Set of preimages $= \{1, 2, 3, 4\}$
Set of images $= \{5, 6, 7, 8\}$

Function Identification

A function can be identified by the following three methods:

- Using mapping
- By rules
- Using vertical line test

By rules

- For every x, there should be a y.
- For every x, there should be only one y.

Mapping method

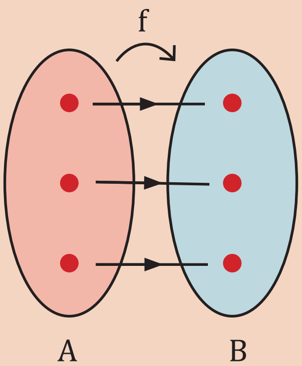


Diagram 1

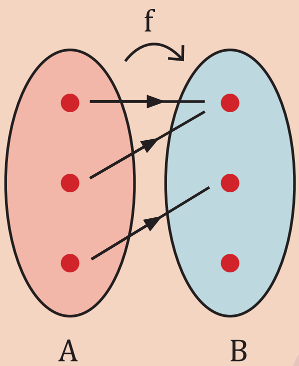


Diagram 2

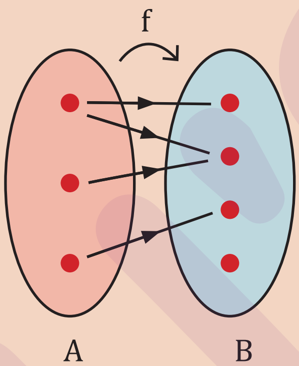


Diagram 3

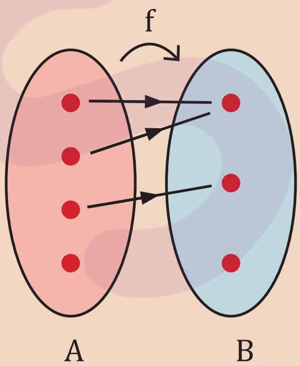


Diagram 4

1. All the inputs have only one output. It is a function.
2. All the inputs have only one output. It is a function.
3. The first element has two outputs. It is not a function.
4. The last element does not have any output. It is not a function.

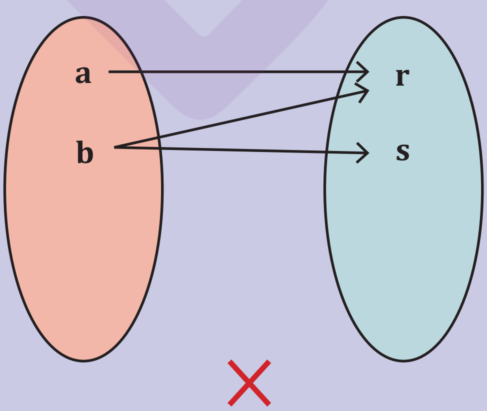
If X = {a, b, c, d, e} and Y = {p, q, r, s, t}, then which of the following subset(s) of X × Y is/are function(s) from X to Y.

(a) {(a, r) (b, r) (b, s) (d, t) (e, q) (c, q)} **(b) {(a, p) (b, t) (c, r) (d, s) (e, q)}**

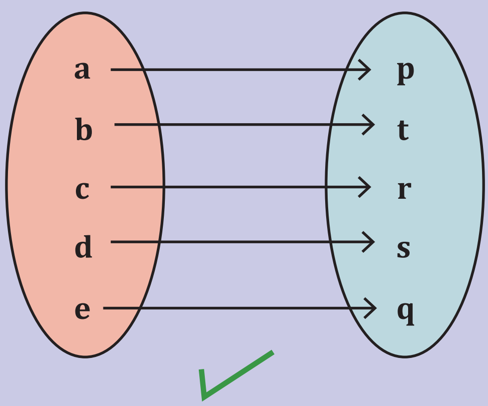
(c) {(a, r) (b, p) (c, t) (d, q)} **(d) {(a, r) (b, r) (c, r) (d, r) (e, r)}**

Solution

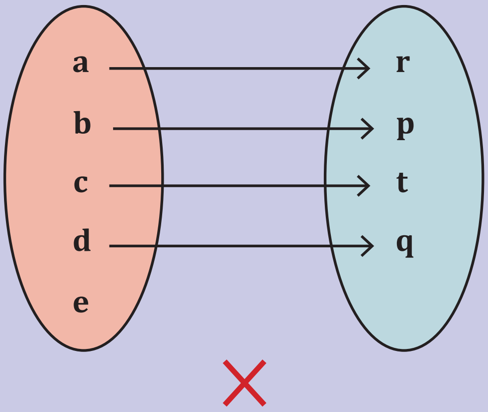
(a)



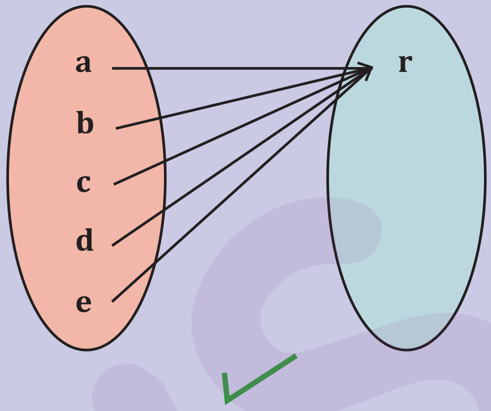
(b)



(c)



(d)



From the diagram, it is clear that,

a. b has two outputs. It is not a function.

b. All the inputs have only one output. It is a function.

c. Element e does not have any output. It is not a function.

d. All the inputs have only one output r. It is a function.

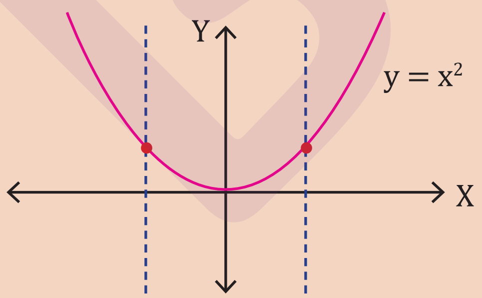
Thus, options (b) and (d) are the correct answers.

Vertical line test

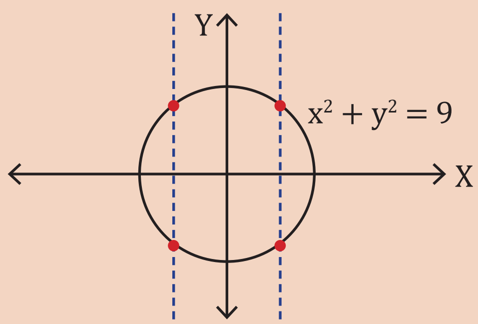
A graph is known to be a function if the vertical line drawn parallel to the Y-axis does not intersect the graph at more than one point in the domain.

Examples

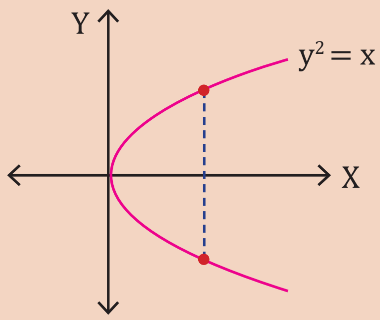
The vertical line intersects the graph of $y = x^2$ at one point only.
Hence, $y = x^2$ is a function.



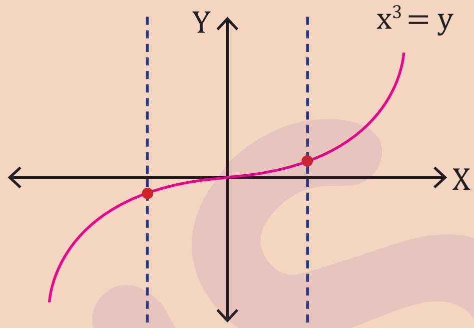
The vertical line intersects the graph of $x^2 + y^2 = 9$ at two points.
Hence, $x^2 + y^2 = 9$ is not a function.



The vertical line intersects the graph of $y^2 = x$ at two points.
Hence, $y^2 = x$ is not a function.



The vertical line intersects the graph of $y = x^3$ at one point only.
Hence, $y = x^3$ is a function.



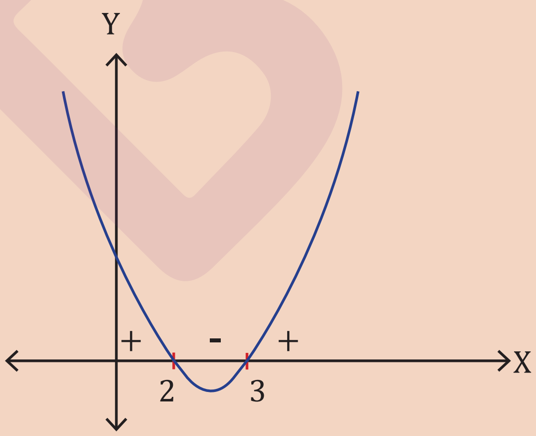
Graphical method

A graph can be made by the following three methods:

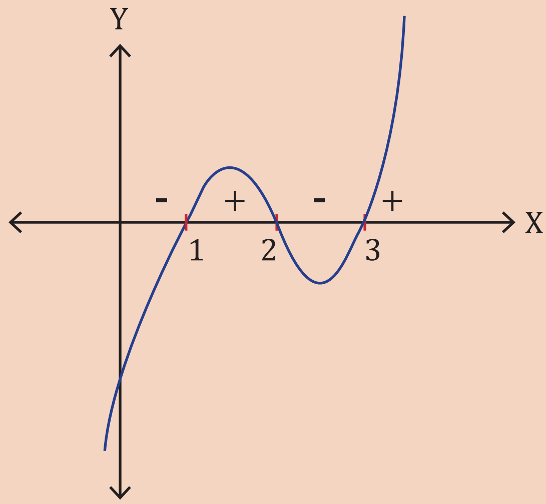
- Using wavy curve method → This method is applicable only in those functions whose factorization is possible.
- Using transformation
- Using calculus

Wavy Curve Method

Draw the graph of $y = x^2 - 5x + 6$
$\Rightarrow y = (x - 2)(x - 3)$

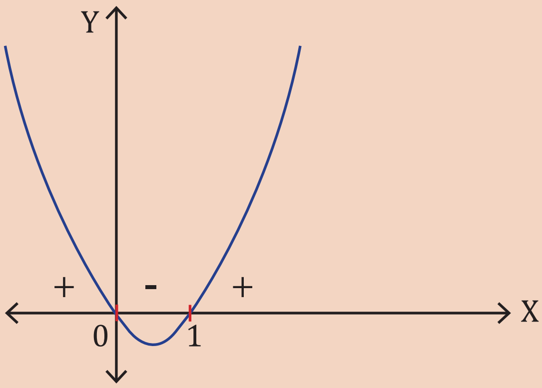


Draw the graph of $y = x^3 - 6x^2 + 11x - 6$
$\Rightarrow y = (x - 1)(x - 2)(x - 3)$

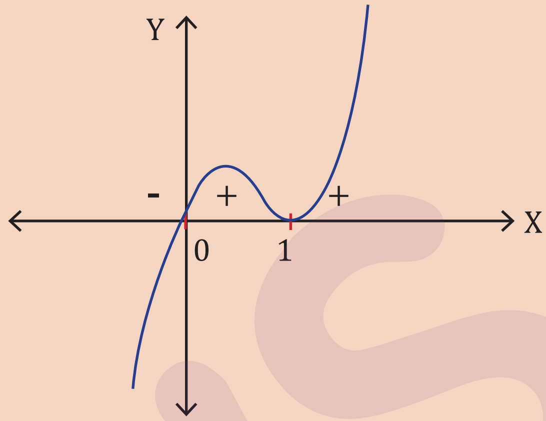


Draw the graph of $y = x^2 - x$

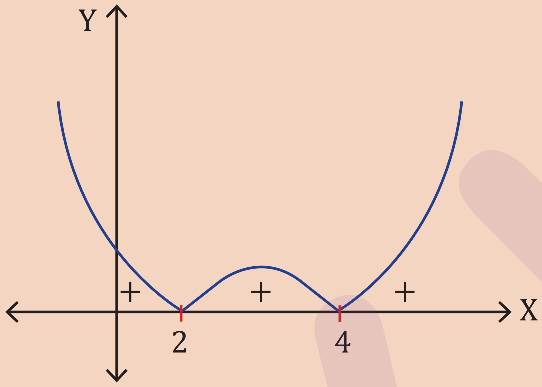
$\Rightarrow y = x(x - 1)$



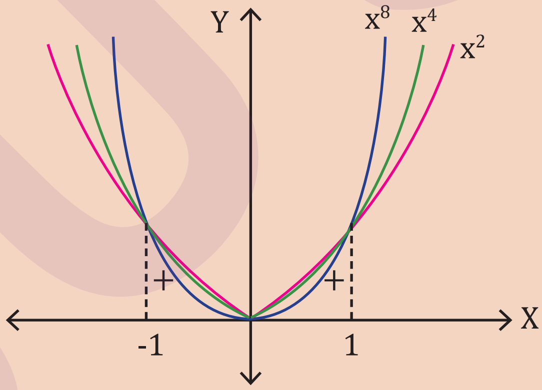
Draw the graph of $y = x(x - 1)^2$



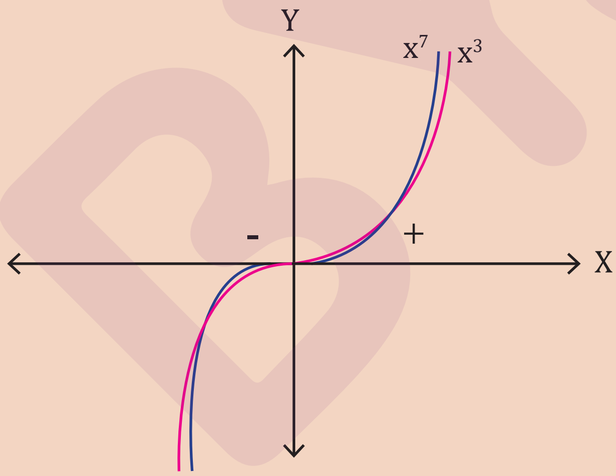
Draw the graph of $y = (x - 2)^2(x - 4)^4$



Draw the graph of $y = x^2, x^4, x^8$

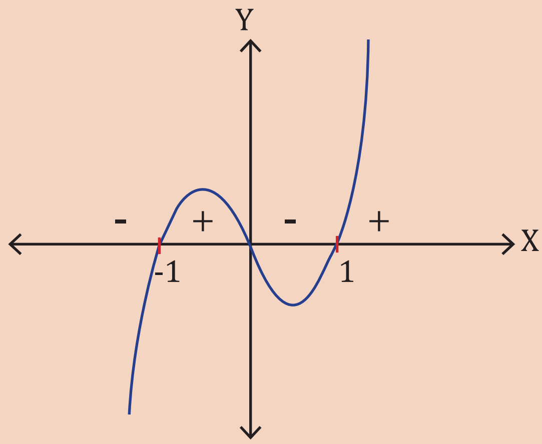


Draw the graph of $y = x^3, x^7$



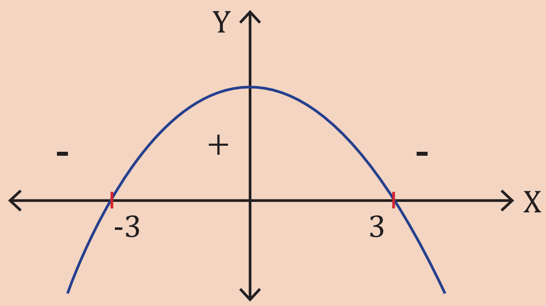
Draw the graph of $y = x^3 - x$

$\Rightarrow y = x(x^2 - 1)$



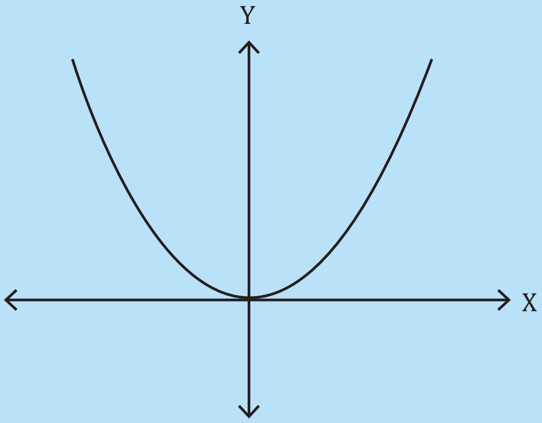
Draw the graph of $y = 9 - x^2$

$y = -(x + 3)(x - 3)$

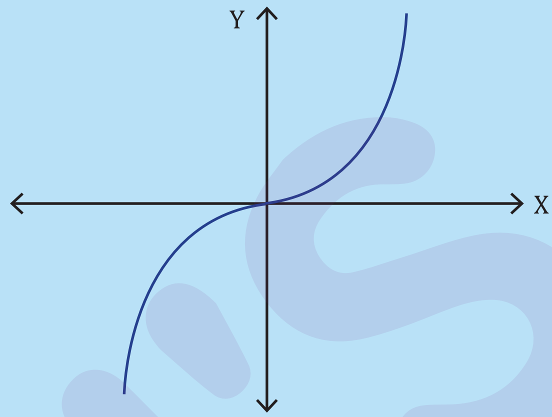


Note

$y = x^{even}$



$y = x^{(2n+1)}$;$n \in \mathbb{N}$



Domain, Co-domain, and Range of a Function

- If function f is defined from a set A to a set B, then for $f : A \rightarrow B$.
  A is known as the **domain** of function f and B is known as the **co-domain** of function f.
  The set of f-images of elements of A is known as the range of function f.

Domain: All the possible values of x for which f(x) exists

Range: All the possible values of f(x) for all the values of x

Note

- Range is also known as a set of images or height of graph.

Function	Domain	Range
Polynomial function	$\mathbb{R}$	$\mathbb{R}$
Identity function	$\mathbb{R}$	$\mathbb{R}$
Constant function	$\mathbb{R}$	{c} (Value of constant)
Reciprocal function	$\mathbb{R}_0$	$\mathbb{R}_0$
Signum function	$\mathbb{R}$	{-1, 0, 1}

$ax + b;\ a, b \in \mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$ax^3 + b;\ a, b \in \mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$x^2, \|x\|$	$\mathbb{R}$	$\mathbb{R}^+ \cup 0$ i.e., $[0, \infty)$
x^3	$\mathbb{R}$	$\mathbb{R}$
$x + \|x\|$	$\mathbb{R}$	$\mathbb{R}^+ \cup 0$ i.e., $[0, \infty)$
$x - \|x\|$	$\mathbb{R}$	$\mathbb{R}^- \cup 0$ i.e., $(-\infty, 0]$
$[x]$	$\mathbb{R}$	$\mathbb{Z}$
$x - [x]$	$\mathbb{R}$	$[0, 1)$
$\frac{\|x\|}{x}$	$\mathbb{R}_0$	$\{-1, 1\}$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$a^x\ (a > 0)$	$\mathbb{R}$	$\mathbb{R}^+$ i.e., $(0, \infty)$
$\log x$	$\mathbb{R}^+$ i.e., $(0, \infty)$	$\mathbb{R}$
$\sin x$	$\mathbb{R}$	$[-1, 1]$
$\cos x$	$\mathbb{R}$	$[-1, 1]$
$\tan x$	$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\};\ n \in \mathbb{Z}$	$\mathbb{R}$
$\cot x$	$\mathbb{R} - \{n\pi\};\ n \in \mathbb{Z}$	$\mathbb{R}$

sec x	$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\}; n \in \mathbb{Z}$	$(-\infty, -1] \cup [1, \infty)$
cosec x	$\mathbb{R} - \{n\pi\}; n \in \mathbb{Z}$	$(-\infty, -1] \cup [1, \infty)$
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Concept Check

Find the domain and range of the following: $f(x) = x^4 + x^2 + 4$

Summary Sheet

Key Takeaways

- A function is a relation defined from set A to set B when it has the following properties:
  (i) Each element of A is associated to some element in B.
  (ii) The association is unique.
- A graph is known to be a function if the vertical line drawn parallel to the Y-axis does not intersect the graph at more than one point in the domain.
- **Domain**: It is the value of set A for which a function is defined.
- **Range**: It consists of all the values that the function gives.
- **Co-domain**: It consists of the set of all the elements in set B. (Range $\subseteq$ Co-domain)

Mind Map

- Functions
  - Definition
  - Identification of function
    - By rules
    - Mapping method
    - Vertical line test
  - Domain, co-domain and range

Self-Assessment

Find the domain and range of $\log(x - 2)$

Answers

Concept Check

Given, $f(x) = x^4 + x^2 + 4$

Since $f(x)$ is a polynomial function, its domain is $\mathbb{R}$.

Let $y = x^4 + x^2 + 4$

$\Rightarrow y = x^4 + x^2 + 4 + \frac{1}{4} - \frac{1}{4}$ (To make it perfect square)

$\Rightarrow y = \left(x^2 + \frac{1}{2}\right)^2 + \frac{15}{4}$

$\Rightarrow y \geq \frac{1}{4} + \frac{15}{4}$

$\Rightarrow y \geq 4$

$\therefore$ Range: $[4, \infty)$

Self Assessment

For a logarithmic function, the argument is always positive.

$\Rightarrow x - 2 > 0$

$\Rightarrow x > 2$

$\therefore$ Domain: $(2, \infty)$

Range: $f(x) \in \mathbb{R}$